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Idempotent Elements in Tricomplex Numbers

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ABSTRACT

In 1892, in search for special algebras, Corrado Segre published a paper in which he introduced an infinite family of algebras whose elements are called bicomplex numbers, tricomplex numbers… , n – complex numbers. In that paper Segre introduced idempotent elements of bicomplex numbers. Idempotent elements play a central role in the theory of bicomplex numbers. This paper introduces the algebraic structure of Tricomplex Numbers exploring some of their fundamental properties. We have identified and characterized sixteen distinct idempotent elements of Tricomplex Numbers. We also discuss their properties and establish the relationships among them.

Keywords: *Bicomplex Numbers, Tricomplex Numbers, Idempotent elements*.

2010 AMS Subject Classification: *30G35, 32A30, 32A10, 13A18.*

1. Introduction

The set of bicomplex numbers is defined as (For detail cf. [**1**], **[2], [3],[4], [5]**.):

$$\mathbb{C}(i_1, i_2) = \{x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 : x_1, x_2, x_3, x_4 \in \mathbb{C}_0\}$$

where $i_1 \neq i_2, i_1^2 = i_2^2 = -1$ and, $i_1 i_2 = i_2 i_1$.

We shall use the notation $\mathbb{C}_0$ for the set of real numbers and $\mathbb{C}(i_1)$, $\mathbb{C}(i_2)$ for the following sets:

$$\mathbb{C}(i_1) = \{x + i_1 y : x, y \in \mathbb{C}_0\}$$

$$\mathbb{C}(i_2) = \{x + i_2 y : x, y \in \mathbb{C}_0\}$$

Analogously, we also define $\mathbb{C}(i_1, i_3)$ and $\mathbb{C}(i_2, i_3)$.

1.1 Idempotent elements in $\mathbb{C}(i_1, i_2)$

Besides 0 and 1, there are exactly two non-trivial idempotent elements exist in $\mathbb{C}(i_1, i_2)$, denoted as e_1 and $e_1^\dagger$ and defined as $e_1 = \frac{1+i_1 i_2}{2}$ and $e_1^\dagger = \frac{1-i_1 i_2}{2}$. Note that $e_1 + e_1^\dagger = 1$ and $e_1 e_1^\dagger = e_1^\dagger e_1 = 0$.

1.2 Idempotent elements in $\mathbb{C}(i_1, i_3)$

Besides 0 and 1, there are exactly two non-trivial idempotent elements exist in $\mathbb{C}(i_1, i_3)$, denoted as e_2 and $e_2^\dagger$ and defined as $e_2 = \frac{1+i_1 i_3}{2}$ and $e_2^\dagger = \frac{1-i_1 i_3}{2}$. Note that $e_2 + e_2^\dagger = 1$ and $e_2 e_2^\dagger = e_2^\dagger e_2 = 0$.

1.3 Idempotent elements in $\mathbb{C}(i_2, i_3)$

Besides 0 and 1, there are exactly two non-trivial idempotent elements exist in $\mathbb{C}(i_2, i_3)$, denoted as e_3 and $e_3^\dagger$ and defined as $e_3 = \frac{1+i_2 i_3}{2}$ and $e_3^\dagger = \frac{1-i_2 i_3}{2}$. Note that $e_3 + e_3^\dagger = 1$ and $e_3 e_3^\dagger = e_3 e_3^\dagger = 0$.

1.4 Idempotent Representation of the elements of $\mathbb{C}(\boldsymbol{i_1}, \boldsymbol{i_2})$

The bicomplex numbers can be represented in two different idempotent forms w.r.t. the elements from $\mathbb{C}(i_1)$ and $\mathbb{C}(i_2)$, explained as follows :

(a) The $\mathbb{C}(i_1)$– idempotent representation of Bicomplex Numbers is given by

$$\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = (x_1 + i_1 x_2) + i_2(x_3 + i_1 x_4)$$

$$= z_1 + i_2 z_2 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_1^\dagger = {}^1\xi e_1 + {}^2\xi e_1^\dagger,$$

where, ${}^1\xi = z_1 - i_1 z_2 = (x_1 + x_4) + i_1(x_2 - x_3)$,

$${}^2\xi = z_1 + i_1 z_2 = (x_1 - x_4) + i_1(x_2 + x_3) \in \mathbb{C}(i_1)$$

(b) The $\mathbb{C}(i_2)$–idempotent representation of Bicomplex Numbers is given by

$$\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = (x_1 + i_2 x_3) + i_1(x_2 + i_2 x_4)$$

$$= w_1 + i_1 w_2 = (w_1 - i_2 w_2)e_1 + (w_1 + i_2 w_2)e_1^\dagger = \xi_1 e_1 + \xi_2 e_1^\dagger,$$

where, $\xi_1 = w_1 - i_2 w_2 = (x_1 + x_4) - i_2(x_2 - x_3)$,

$$\xi_2 = w_1 + i_2 w_2 = (x_1 - x_4) + i_2(x_2 + x_3) \in \mathbb{C}(i_2)$$

1.5 Singular Elements in $\mathbb{C}(\boldsymbol{i_1}, \boldsymbol{i_2})$*:*

Let $\xi, \eta \in \mathbb{C}(i_1, i_2)$ such that $\xi\eta = \eta\xi = 1$, then η is said to be a multiplicative inverse of ξ. The invertible elements are also called non-singular elements. The set of all singular elements in $\mathbb{C}(i_1, i_2)$ is denoted as $\mathbb{O}(i_1, i_2)$ and $\mathbb{C}(i_1, i_2)\backslash\mathbb{O}(i_1, i_2)$ is the set of all non-singular elements in $\mathbb{C}(i_1, i_2)$.

2. The Set of Tricomplex Numbers

The set of Tricomplex Numbers is defined as:

$$\mathbb{C}_3 = \mathbb{C}(i_1, i_2, i_3)$$

$$= \{x_1 + i_1 x_2 + i_2 x_3 + i_3 x_4 + i_1 i_2 x_5 + i_1 i_3 x_6 + i_2 i_3 x_7 + i_1 i_2 i_3 x_8 : x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \in \mathbb{C}_0\}$$

Where $i_1 \neq i_2 \neq i_3$; $i_1^2 = i_2^2 = i_3^2 = -1$, and $i_1 i_2 = i_2 i_1, i_1 i_3 = i_3 i_1, i_2 i_3 = i_3 i_2$.

2.1 Idempotent Elements in $\mathbb{C}(i_1, i_2, i_3)$ *:*

Let $\zeta = \xi + i_3\eta \in \mathbb{C}(i_1, i_2, i_3)$; $\xi, \eta \in \mathbb{C}(i_1, i_2)$ be an idempotent element

$$\Rightarrow \zeta^2 = \zeta$$

$$\Rightarrow (\xi + i_3\eta)^2 = \xi + i_3\eta$$

$\Rightarrow \xi^2 - \eta^2 + i_3 2\xi\eta = \xi + i_3\eta$

$\Rightarrow \xi^2 - \eta^2 = \xi$ and $2\xi\eta = \eta$ ………………..**(i)**

Now for $\eta \in \mathbb{C}(i_1, i_2)$, either $\eta \notin \mathbb{O}(i_1, i_2)$ or $\eta \in \mathbb{O}(i_1, i_2)$

Case (i) $\eta \notin \mathbb{O}(i_1, i_2)$

From equation **(i)**

$\xi^2 - \eta^2 = \xi$ and $2\xi\eta = \eta$

Let $2\xi\eta = \eta$

$\Rightarrow 2\xi = 1$ $\quad (\because \eta \notin \mathbb{O}(i_1, i_2))$

$\Rightarrow \xi = \frac{1}{2}$

From $\xi^2 - \eta^2 = \xi$

$\Rightarrow \eta^2 = \xi^2 - \xi = \left(\frac{1}{2}\right)^2 - \frac{1}{2}$

$\Rightarrow \eta^2 = -\frac{1}{4}$

$\Rightarrow ({}^1\eta)^2 e_1 + ({}^2\eta)^2 e_1^\dagger = -\frac{1}{4} e_1 - \frac{1}{4} e_1^\dagger$

$\Rightarrow ({}^1\eta)^2 = -\frac{1}{4}$ and $({}^2\eta)^2 = -\frac{1}{4}$

$\Rightarrow {}^1\eta = \frac{i_1}{2}, -\frac{i_1}{2}$ and ${}^2\eta = \frac{i_1}{2}, -\frac{i_1}{2}$

$\Rightarrow \eta = {}^1\eta\, e_1 + {}^2\eta\, e_1^\dagger$

$\Rightarrow \eta = \frac{i_1}{2}, -\frac{i_1}{2}, \frac{i_2}{2}, -\frac{i_2}{2}$

So, we get $\xi = \frac{1}{2}$ and $\eta = \frac{i_1}{2}, -\frac{i_1}{2}, \frac{i_2}{2}, -\frac{i_2}{2}$

Now,

$\zeta = \xi + i_3\eta$

$\Rightarrow \zeta = e_2 = \frac{1 + i_1 i_3}{2}, e_2^\dagger = \frac{1 - i_1 i_3}{2}, e_3 = \frac{1 + i_2 i_3}{2}, e_3^\dagger = \frac{1 - i_2 i_3}{2}$

Case (ii) $\eta \in \mathbb{O}(i_1, i_2)$

Then $\eta = 0$ or αe_1 or $\beta e_1^\dagger; \alpha \neq 0, \beta \neq 0 \in \mathbb{C}(i_1)$

(a) $\eta = 0$

From equation **(i)**

$\xi^2 - \eta^2 = \xi$ and $2\xi\eta = \eta$

$\Rightarrow \xi^2 = \xi$

$\Rightarrow \xi = 0,1, e_1, e_1^\dagger$

Now, $\xi = 0,1, e_1, e_1^\dagger$ and $\eta = 0$

$\zeta = \xi + i_3\eta$

$\Rightarrow \zeta = 0,1, e_1, e_1^\dagger$

(b) $\eta = \alpha e_1; \alpha \neq 0$

From **(i)**

$\xi^2 - \eta^2 = \xi$ and $2\xi\eta = \eta$

$2\xi\eta = \eta$

$\Rightarrow 2\xi\alpha e_1 = \alpha e_1$

$\Rightarrow 2\,{}^1\xi\alpha e_1 = \alpha e_1$

$\Rightarrow 2\,{}^1\xi e_1 = e_1$

$\Rightarrow 2\,{}^1\xi = 1$

$\Rightarrow {}^1\xi = \frac{1}{2}$

Now, from

$\xi^2 - \eta^2 = \xi$

$\Rightarrow ({}^1\xi)^2 e_1 + ({}^2\xi)^2 e_1^\dagger - \alpha^2 e_1 = {}^1\xi\, e_1 + {}^2\xi e_1^\dagger$

$\Rightarrow \frac{1}{4} e_1 + ({}^2\xi)^2 e_1^\dagger - \alpha^2 e_1 = \frac{1}{2}\, e_1 + {}^2\xi\, e_1^\dagger$

$\Rightarrow \left(\frac{1}{4} - \alpha^2\right) e_1 + ({}^2\xi)^2 e_1^\dagger = \frac{1}{2}\, e_1 + {}^2\xi\, e_1^\dagger$

$\Rightarrow \frac{1}{4} - \alpha^2 = \frac{1}{2}$ and $({}^2\xi)^2 = {}^2\xi$

$\Rightarrow \alpha = \frac{i_1}{2}, -\frac{i_1}{2}$ and ${}^2\xi = 0,1$

$\Rightarrow \alpha = \frac{i_1}{2}, -\frac{i_1}{2}$ and ${}^2\xi = 0,1$

$\xi = {}^1\xi\, e_1 + {}^2\xi\, e_1^\dagger = \frac{1}{2} e_1, \frac{1}{2} e_1 + e_1^\dagger$ and $\eta = \alpha e_1 = \frac{i_1}{2}\, e_1, -\frac{i_1}{2}\, e_1$

Now,

$\zeta = \xi + i_3\eta$

$\Rightarrow \zeta = \frac{1}{2}e_1 + i_3\frac{i_1}{2}\ e_1 = \left(\frac{1 + i_1i_3}{2}\right)e_1 = e_1e_2,$

$\zeta = \frac{1}{2}e_1 - i_3\frac{i_1}{2}\ e_1 = \left(\frac{1 - i_1i_3}{2}\right)e_1 = e_1e_2^{\dagger},$

$\zeta = \left(\frac{1}{2}e_1 + e_1^{\dagger}\right) + i_3\frac{i_1}{2}\ e_1 = e_1^{\dagger} + \left(\frac{1 + i_1i_3}{2}\right)e_1 = e_1^{\dagger} + e_1e_2 = 1 - e_1e_2^{\dagger},$

$\zeta = \left(\frac{1}{2}e_1 + e_1^{\dagger}\right) - i_3\frac{i_1}{2}\ e_1 = e_1^{\dagger} + \left(\frac{1 - i_1i_3}{2}\right)e_1 = e_1^{\dagger} + e_2^{\dagger}e_1 = 1 - e_1e_2$

Hence,

$\zeta = e_1e_2, e_1e_2^{\dagger}, 1 - e_1e_2^{\dagger}, 1 - e_1e_2$

(c) $\eta = \beta e_1^{\dagger}; \beta \neq 0$

From **(i)**

$\xi^2 - \eta^2 = \xi$ and $2\xi\eta = \eta$

$2\xi\eta = \eta$

$\Rightarrow 2\xi\beta e_1^{\dagger} = \beta e_1^{\dagger}$

$\Rightarrow 2\,{}^2\xi\beta e_1^{\dagger} = \beta e_1^{\dagger}$

$\Rightarrow 2\,{}^2\xi e_1^{\dagger} = e_1^{\dagger}$

$\Rightarrow 2\,{}^2\xi = 1$

$\Rightarrow {}^2\xi = \frac{1}{2}$

Now, from

$\xi^2 - \eta^2 = \xi$

$\Rightarrow ({}^1\xi)^2e_1 + ({}^2\xi)^2e_1^{\dagger} - \beta^2e_1^{\dagger} = {}^1\xi\ e_1 + {}^2\xi\ e_1^{\dagger}$

$\Rightarrow ({}^1\xi)^2e_1 + \frac{1}{4}e_1^{\dagger} - \beta^2e_1^{\dagger} = {}^1\xi\ e_1 + \frac{1}{2}e_1^{\dagger}$

$\Rightarrow ({}^1\xi)^2e_1 + \left(\frac{1}{4} - \beta^2\right)e_1^{\dagger} = {}^1\xi\ e_1 + \frac{1}{2}e_1^{\dagger}$

$\Rightarrow ({}^1\xi)^2 = {}^1\xi$ and $\frac{1}{4} - \beta^2 = \frac{1}{2}$

$\Rightarrow {}^1\xi = 0,1$ and $\beta = \frac{i_1}{2}, -\frac{i_1}{2}$

$\xi = {}^1\xi\ e_1 + {}^2\xi\ e_1^{\dagger} = \frac{1}{2}e_1^{\dagger}, e_1 + \frac{1}{2}e_1^{\dagger}$ and $\eta = \beta e_1^{\dagger} = \frac{i_1}{2}\ e_1^{\dagger}, -\frac{i_1}{2}\ e_1^{\dagger}$

Now, $\zeta = \xi + i_3\eta$

$$\zeta = \frac{1}{2}e_1^\dagger + i_3\frac{i_1}{2}\ e_1^\dagger = \left(\frac{1 + i_1 i_3}{2}\right)e_1^\dagger = e_1^\dagger e_2,$$

$$\zeta = \frac{1}{2}e_1^\dagger - i_3\frac{i_1}{2}\ e_1^\dagger = \left(\frac{1 - i_1 i_3}{2}\right)e_1^\dagger = e_1^\dagger e_2^\dagger,$$

$$\zeta = \left(e_1^\dagger + \frac{1}{2}e_1^\dagger\right) + i_3\frac{i_1}{2}\ e_1^\dagger = e_1 + \left(\frac{1 + i_1 i_3}{2}\right)e_1^\dagger = e_1 + e_1^\dagger e_2 = 1 - e_1^\dagger e_2^\dagger,$$

$$\zeta = \left(e_1 + \frac{1}{2}e_1^\dagger\right) - i_3\frac{i_1}{2}\ e_1^\dagger = e_1 + \left(\frac{1 - i_1 i_3}{2}\right)e_1^\dagger = e_1 + e_1^\dagger e_2^\dagger = 1 - e_1^\dagger e_2$$

Hence,

$$\zeta = e_1^\dagger e_2, e_1^\dagger e_2^\dagger, 1 - e_1^\dagger e_2^\dagger, 1 - e_1^\dagger e_2$$

We get total 16 idempotent elements which are

$$0,1, e_1, e_1^\dagger, e_2, e_2^\dagger, e_3, e_3^\dagger, e_1e_2, e_1e_2^\dagger, 1 - e_1e_2^\dagger, 1 - e_1e_2, e_1^\dagger e_2, e_1^\dagger e_2^\dagger, 1 - e_1^\dagger e_2^\dagger, 1 - e_1^\dagger e_2$$

2.2 Different Idempotent Representation of Tricomplex Number:

There are several idempotent representations possible for Tricomplex numbers. Some of them are shown here.

(R1) A Tricomplex number $\zeta = \xi + i_3\,\eta;\ \xi, \eta \in \mathbb{C}(i_1, i_2)$ can be represent as

$$\xi + i_3\,\eta = (\xi - i_1\,\eta)\left(\frac{1 + i_1 i_3}{2}\right) + (\xi + i_1\,\eta)\left(\frac{1 - i_1 i_3}{2}\right) = (\xi - i_1\,\eta)e_2 + (\xi + i_1\,\eta)e_2^\dagger$$

$$\xi + i_3\,\eta = (\xi - i_2\,\eta)\left(\frac{1 + i_2 i_3}{2}\right) + (\xi + i_2\,\eta)\left(\frac{1 - i_2 i_3}{2}\right) = (\xi - i_2\,\eta)e_3 + (\xi + i_2\,\eta)e_3^\dagger$$

(R2) A Tricomplex number $\zeta = \xi + i_2\,\eta;\ \xi, \eta \in \mathbb{C}(i_1, i_3)$ can be represent as

$$\xi + i_2\,\eta = (\xi - i_1\,\eta)\left(\frac{1 + i_1 i_2}{2}\right) + (\xi + i_1\,\eta)\left(\frac{1 - i_1 i_2}{2}\right) = (\xi - i_1\,\eta)e_1 + (\xi + i_1\,\eta)e_1^\dagger$$

$$\xi + i_2\,\eta = (\xi - i_3\,\eta)\left(\frac{1 + i_2 i_3}{2}\right) + (\xi + i_3\,\eta)\left(\frac{1 - i_2 i_3}{2}\right) = (\xi - i_3\,\eta)e_3 + (\xi + i_3\,\eta)e_3^\dagger$$

(R3) A Tricomplex number $\zeta = \xi + i_1\,\eta;\ \xi, \eta \in \mathbb{C}(i_2, i_3)$ can be represent as

$$\xi + i_1\,\eta = (\xi - i_2\,\eta)\left(\frac{1 + i_1 i_2}{2}\right) + (\xi + i_2\,\eta)\left(\frac{1 - i_1 i_2}{2}\right) = (\xi - i_2\,\eta)e_1 + (\xi + i_2\,\eta)e_1^\dagger$$

$$\xi + i_1\,\eta = (\xi - i_3\,\eta)\left(\frac{1 + i_2 i_3}{2}\right) + (\xi + i_3\,\eta)\left(\frac{1 - i_2 i_3}{2}\right) = (\xi - i_3\,\eta)e_3 + (\xi + i_3\,\eta)e_3^\dagger$$

2.3 Idempotent Elements by using idempotent techniques:

Let $\zeta = \xi e_1 + \eta e_1^\dagger \in \mathbb{C}(i_1, i_2, i_3)$; $\xi, \eta \in \mathbb{C}(i_1, i_3)$ be an idempotent element

$$\Rightarrow \zeta^2 = \zeta$$

$$\Rightarrow \left(\xi e_1 + \eta e_1^\dagger\right)^2 = \xi e_1 + \eta e_1^\dagger$$

$\Rightarrow \xi^2 e_1 + \eta^2 e_1^\dagger = \xi e_1 + \eta e_1^\dagger$

$\Rightarrow \xi^2 = \xi$ and $\eta^2 = \eta$

$\Rightarrow \xi = 0,1, e_2, e_2^\dagger$ and $\eta = 0,1, e_2, e_2^\dagger$

As, $\zeta = \xi e_1 + \eta e_1^\dagger$, therefore

1	$\xi = 0$	$\eta = 0$	$\zeta = 0e_1 + 0e_1^\dagger = 0$
2	$\xi = 0$	$\eta = 1$	$\zeta = 0e_1 + 1e_1^\dagger = e_1^\dagger$
3	$\xi = 0$	$\eta = e_2$	$\zeta = 0e_1 + e_2 e_1^\dagger = e_1^\dagger e_2$
4	$\xi = 0$	$\eta = e_2^\dagger$	$\zeta = 0e_1 + e_2^\dagger e_1^\dagger = e_1^\dagger e_2^\dagger$
5	$\xi = 1$	$\eta = 0$	$\zeta = 1e_1 + 0e_1^\dagger = e_1$
6	$\xi = 1$	$\eta = 1$	$\zeta = 1e_1 + 1e_1^\dagger = 1$
7	$\xi = 1$	$\eta = e_2$	$\zeta = 1e_1 + e_2 e_1^\dagger = e_1 + e_2 e_1^\dagger = 1 - e_1^\dagger e_2^\dagger$
8	$\xi = 1$	$\eta = e_2^\dagger$	$\zeta = 1e_1 + e_2^\dagger e_1^\dagger = e_1 + e_2^\dagger e_1^\dagger = 1 - e_1^\dagger e_2$
9	$\xi = e_2$	$\eta = 0$	$\zeta = \mathrm{e}_2 e_1 + 0e_1^\dagger = e_1 e_2$
10	$\xi = e_2$	$\eta = 1$	$\zeta = \mathrm{e}_2 e_1 + 1e_1^\dagger = e_1 e_2 + e_1^\dagger = 1 - e_1 e_2^\dagger$
11	$\xi = e_2$	$\eta = e_2$	$\zeta = \mathrm{e}_2 e_1 + e_2 e_1^\dagger = e_2$
12	$\xi = e_2$	$\eta = e_2^\dagger$	$\zeta = \mathrm{e}_2 e_1 + e_2^\dagger e_1^\dagger = e_3^\dagger$
13	$\xi = e_2^\dagger$	$\eta = 0$	$\zeta = e_2^\dagger e_1 + 0e_1^\dagger = e_1 e_2^\dagger$
14	$\xi = e_2^\dagger$	$\eta = 1$	$\zeta = e_2^\dagger e_1 + 1e_1^\dagger = e_2^\dagger e_1 + e_1^\dagger = 1 - e_1 e_2$
15	$\xi = e_2^\dagger$	$\eta = e_2$	$\zeta = e_2^\dagger e_1 + e_2 e_1^\dagger = e_3$
16	$\xi = e_2^\dagger$	$\eta = e_2^\dagger$	$\zeta = e_2^\dagger e_1 + e_1^\dagger e_2^\dagger = e_2^\dagger$

2.4 Idempotent Elements in $\mathbb{C}(i_1, i_2, i_3)$ by using idempotent techniques in different idempotent representation

Let $\zeta = \xi e_2 + \eta e_2^\dagger \in \mathbb{C}(i_1, i_2, i_3)$; $\xi, \eta \in \mathbb{C}(i_1, i_2)$ be an idempotent element

$\Rightarrow \zeta^2 = \zeta$

$\Rightarrow \left(\xi e_2 + \eta e_2^\dagger\right)^2 = \xi e_2 + \eta e_2^\dagger$

$\Rightarrow \xi^2 e_2 + \eta^2 e_2^\dagger = \xi e_2 + \eta e_2^\dagger$

$\Rightarrow \xi^2 = \xi$ and $\eta^2 = \eta$

$\Rightarrow \xi = 0,1, e_1, e_1^\dagger$ and $\eta = 0,1, e_1, e_1^\dagger$

As, $\zeta = \xi e_2 + \eta e_2^\dagger$, therefore

1	$\xi = 0$	$\eta = 0$	$\zeta = 0e_2 + 0e_2^\dagger = 0$
2	$\xi = 0$	$\eta = 1$	$\zeta = 0e_2 + 1e_2^\dagger = e_2^\dagger$
3	$\xi = 0$	$\eta = e_1$	$\zeta = 0e_2 + e_1 e_2^\dagger = e_1 e_2^\dagger$
4	$\xi = 0$	$\eta = e_1^\dagger$	$\zeta = 0e_2 + e_1^\dagger e_2^\dagger = e_1^\dagger e_2^\dagger$
5	$\xi = 1$	$\eta = 0$	$\zeta = 1e_2 + 0e_2^\dagger = e_2$
6	$\xi = 1$	$\eta = 1$	$\zeta = 1e_2 + 1e_2^\dagger = 1$
7	$\xi = 1$	$\eta = e_1$	$\zeta = 1e_2 + e_1 e_2^\dagger = e_2 + e_1 e_2^\dagger = 1 - e_1^\dagger e_2^\dagger$
8	$\xi = 1$	$\eta = e_1^\dagger$	$\zeta = 1e_2 + e_1^\dagger e_2^\dagger = e_2 + e_1^\dagger e_2^\dagger = 1 - e_1 e_2^\dagger$
9	$\xi = e_1$	$\eta = 0$	$\zeta = \mathrm{e}_1 e_2 + 0e_2^\dagger = e_1 e_2$
10	$\xi = e_1$	$\eta = 1$	$\zeta = \mathrm{e}_1 e_2 + 1e_2^\dagger = e_1 e_2 + e_2^\dagger = 1 - e_1^\dagger e_2$
11	$\xi = e_1$	$\eta = e_1$	$\zeta = \mathrm{e}_1 e_2 + e_1 e_2^\dagger = e_1$
12	$\xi = e_1$	$\eta = e_1^\dagger$	$\zeta = \mathrm{e}_1 e_2 + e_1^\dagger e_2^\dagger = e_3^\dagger$
13	$\xi = e_1^\dagger$	$\eta = 0$	$\zeta = e_1^\dagger e_2 + 0e_2^\dagger = e_1^\dagger e_2$
14	$\xi = e_1^\dagger$	$\eta = 1$	$\zeta = e_1^\dagger e_2 + 1e_2^\dagger = e_1^\dagger e_2 + e_2^\dagger = 1 - e_1 e_2$
15	$\xi = e_1^\dagger$	$\eta = e_1$	$\zeta = e_1^\dagger e_2 + e_1 e_2^\dagger = e_3$
16	$\xi = e_1^\dagger$	$\eta = e_1^\dagger$	$\zeta = e_1^\dagger e_2 + e_1^\dagger e_2^\dagger = e_1^\dagger$

2.5 Idempotent elements and their representation

There are 16 total number of idempotent elements exist in $\mathbb{C}(i_1, i_2, i_3)$.

SN	Notation of idempotent element	Value of idempotent element
1	*0*	*0*
2	*1*	*1*
3	e_1	$\frac{1 + i_1 i_2}{2}$
4	$e_1^\dagger$	$\frac{1 - i_1 i_2}{2}$

SN	Notation of idempotent element	Value of idempotent element
5	e_2	$\frac{1+i_1i_3}{2}$
6	$e_2^\dagger$	$\frac{1-i_1i_3}{2}$
7	e_3	$\frac{1+i_2i_3}{2}$
8	$e_3^\dagger$	$\frac{1-i_2i_3}{2}$
9	$e_4 = e_1e_2$	$\frac{1}{4}(1+i_1i_2+i_1i_3-i_2i_3)$
10	$e_4^\dagger = 1-e_1e_2$	$\frac{1}{4}(3-i_1i_2-i_1i_3+i_2i_3)$
11	$e_5 = e_1e_2^\dagger$	$\frac{1}{4}(1+i_1i_2-i_1i_3+i_2i_3)$
12	$e_5^\dagger = 1-e_1e_2^\dagger$	$\frac{1}{4}(3-i_1i_2+i_1i_3-i_2i_3)$
13	$e_6 = e_1^\dagger e_2$	$\frac{1}{4}(1-i_1i_2+i_1i_3+i_2i_3)$
14	$e_6^\dagger = 1-e_1^\dagger e_2$	$\frac{1}{4}(3+i_1i_2-i_1i_3-i_2i_3)$
15	$e_7 = e_1^\dagger e_2^\dagger$	$\frac{1}{4}(1-i_1i_2-i_1i_3-i_2i_3)$
16	$e_7^\dagger = 1-e_1^\dagger e_2^\dagger$	$\frac{1}{4}(3+i_1i_2+i_1i_3+i_2i_3)$

2.6 Relation between idempotent elements (In the form of multiplication)

(i) $e_1e_1^\dagger = e_1e_6 = e_1e_7 = e_1^\dagger e_4 = e_1^\dagger e_5 = e_2e_2^\dagger = e_2e_5 = e_2e_7 = e_2^\dagger e_4 = e_2^\dagger e_6 = e_3e_3^\dagger = e_3e_4 = e_3e_7 = e_3^\dagger e_5 = e_3^\dagger e_6 = e_4e_4^\dagger = e_4e_5 = e_4e_6 = e_4e_7 = e_5e_5^\dagger = e_5e_6 = e_5e_7 = e_6e_6^\dagger = e_6e_7 = e_7e_7^\dagger = e_1e_2e_3 = 0$

(ii) $e_1e_6^\dagger = e_1e_7^\dagger = e_6^\dagger e_7^\dagger = e_1$

(iii) $e_1^\dagger e_4^\dagger = e_1^\dagger e_5^\dagger = e_4^\dagger e_5^\dagger = e_1^\dagger$

(iv) $e_2e_5^\dagger = e_2e_7^\dagger = e_5^\dagger e_7^\dagger = e_2$

(v) $e_2^\dagger e_4^\dagger = e_2^\dagger e_6^\dagger = e_4^\dagger e_6^\dagger = e_2^\dagger$

(vi) $e_3e_4^\dagger = e_3e_7^\dagger = e_4^\dagger e_7^\dagger = e_3$

(vii) $e_3^\dagger e_5^\dagger = e_3^\dagger e_6^\dagger = e_5^\dagger e_6^\dagger = e_3^\dagger$

(viii) $e_1e_2 = e_1e_3^\dagger = e_1e_4 = e_1e_5^\dagger = e_2e_3^\dagger = e_2e_4 = e_2e_6^\dagger = e_3^\dagger e_4 = e_3^\dagger e_7^\dagger = e_4e_5^\dagger = e_4e_6^\dagger = e_4e_7^\dagger = e_4$

(ix) $e_1e_2^\dagger = e_1e_3 = e_1e_4^\dagger = e_1e_5 = e_2^\dagger e_3 = e_2^\dagger e_5 = e_2^\dagger e_7^\dagger = e_3e_5 = e_3e_6^\dagger = e_4^\dagger e_5 = e_5e_6^\dagger = e_5e_7^\dagger = e_5$

(x) $e_1^\dagger e_2 = e_1^\dagger e_3 = e_1^\dagger e_6 = e_1^\dagger e_7^\dagger = e_2e_3 = e_2e_4^\dagger = e_2e_6 = e_3e_5^\dagger = e_3e_6 = e_4^\dagger e_6 = e_5^\dagger e_6 =$
$e_6e_7^\dagger = e_6$

(xi) $e_1^\dagger e_2^\dagger = e_1^\dagger e_3^\dagger = e_1^\dagger e_6^\dagger = e_1^\dagger e_7 = e_2^\dagger e_3^\dagger = e_2^\dagger e_5^\dagger = e_2^\dagger e_7 = e_3^\dagger e_4^\dagger = e_3^\dagger e_7 = e_4^\dagger e_7 = e_5^\dagger e_7 = e_6^\dagger e_7 = e_7$

2.7 Relation between idempotent elements (In the form of addition)

(i) $e_1 + e_1^\dagger = 1$

(ii) $e_2 + e_2^\dagger = 1$

(iii) $e_3 + e_3^\dagger = 1$

(iv) $e_4 + e_4^\dagger = 1$

(v) $e_5 + e_5^\dagger = 1$

(vi) $e_6 + e_6^\dagger = 1$

(vii) $e_7 + e_7^\dagger = 1$

(viii) $e_1 = e_4 + e_5$

(ix) $e_2 = e_4 + e_6$

(x) $e_3 = e_5 + e_6$

(xi) $e_1^\dagger = e_6 + e_7$

(xii) $e_2^\dagger = e_5 + e_7$

(xiii) $e_3^\dagger = e_4 + e_7$

(xiv) $e_1 + e_6 + e_7 = 1$

(xv) $e_2 + e_5 + e_7 = 1$

(xvi) $e_3 + e_4 + e_7 = 1$

(xvii) $e_1 + e_6 = e_2 + e_5 = e_3 + e_4 = e_7^\dagger$

(xviii) $e_1 + e_7 = e_2^\dagger + e_4 = e_3^\dagger + e_5 = e_6^\dagger$

(xix) $e_1^\dagger + e_4 = e_2 + e_7 = e_3^\dagger + e_6 = e_5^\dagger$

(xx) $e_1^\dagger + e_5 = e_2^\dagger + e_6 = e_3 + e_7 = e_4^\dagger$

2.8 Note :

(i) The set of idempotent elements $E = \{0,1, e_1, e_1^\dagger, e_2, e_2^\dagger, e_3, e_3^\dagger, e_4, e_4^\dagger e_5, e_5^\dagger, e_6, e_6^\dagger, e_7, e_7^\dagger\}$ of $\mathbb{C}(i_1, i_2, i_3)$ is a commutative monoid.

(ii) Total number of idempotent element in $\mathbb{C}(i_1) = 2^{2^0}$

Total number of idempotent element in $\mathbb{C}(i_2) = 2^{2^0}$

Total number of idempotent element in $\mathbb{C}(i_3) = 2^{2^0}$

(iii) Total number of idempotent element in $\mathbb{C}(i_1, i_2) = 2^{2^1}$

Total number of idempotent element in $\mathbb{C}(i_1, i_3) = 2^{2^1}$

Total number of idempotent element in $\mathbb{C}(i_2, i_3) = 2^{2^1}$

(iv) Total number of idempotent element in $\mathbb{C}(i_1, i_2, i_3) = 2^{2^2}$

(v) Total number of idempotent element in $\mathbb{C}(i_1, i_2, i_3, \ldots \ldots \ldots i_n) = 2^{2^{n-1}}$

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