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The Mathematics of Borrowing Wisely: How Governments Optimise Debt

Devagya Suresh Shukla

Grade XII Commerce with Mathematics student at Delhi Public School, Ranipur, India.

ABSTRACT :

Government debt is usually discussed as a number to be minimised, but a growing body of mathematical and economic research instead treats it as a variable to be optimised: a quantity managed against explicit trade-offs between borrowing cost, financing risk, and economic growth. This review draws together three distinct strands of that research: debt sustainability analysis, as used by the International Monetary Fund (IMF); stochastic optimal control theory, as applied by mathematicians studying debt-reduction policy; and portfolio-based debt management, as practised by sovereign debt offices such as the United Kingdom's Debt Management Office (DMO). It explains the mathematics underlying each approach and then examines a well-documented case, the 2010 Reinhart-Rogoff debt-threshold controversy, to show where such models can mislead policy if their assumptions are not scrutinised. The review concludes that mathematical optimisation does not identify a single "correct" debt level for a government, but provides a disciplined framework for making trade-offs between cost, risk, and uncertainty explicit. Given that the IMF projects global public debt to exceed 100 per cent of world GDP by 2029, understanding these frameworks has direct relevance to current fiscal policy debates worldwide.

Introduction

In October 2025, the IMF's Director of Fiscal Affairs, Vitor Gaspar, announced that global public debt was on track to exceed 100 per cent of world GDP by 2029, a level not seen since 1948, in the aftermath of the Second World War.[1] By April 2026, the IMF's Fiscal Monitor was already reporting global public debt at just under 94 per cent of GDP, with risks skewed toward an even faster build-up.[2] Total global debt, public and private combined, stood above 235 per cent of world GDP, with public debt alone reaching roughly $99.2 trillion.[3] Among the G20, seven economies, Canada, China, France, Italy, Japan, the United Kingdom, and the United States, already carry public debt above 100 per cent of GDP.[1]

The instinctive response to numbers like these is to ask how debt can be brought down. That question is under-specified. A government could reduce its debt by defaulting, by inflating it away, by cutting growth through austerity, or by patiently running small surpluses for decades. Each path arrives at a lower debt figure, but each has very different consequences for citizens, financial markets, and future governments.

This is the gap that mathematical optimisation fills. Rather than asking how much debt is "too much" as a moral or political question, it asks: given a government's objectives, such as minimising borrowing costs, avoiding a refinancing crisis, or keeping the probability of default below a set threshold, what is the best achievable path for debt, and what is the best achievable structure of that debt? These are optimisation problems in the strict mathematical sense: an objective function to minimise or maximise, subject to constraints.

This review works through three frameworks that finance ministries, central banks, and mathematicians currently use to answer that question, before examining a case in which one influential debt model was shown to be flawed, and what that reveals about the limits of the approach as a whole.

Frameworks for Optimising Government Debt

Foundations: the debt dynamics equation

Nearly every model of sovereign debt rests on one accounting identity describing how the debt-to-GDP ratio changes from one year to the next:

$$\Delta(D/Y) = [(r - g) / (1 + g)] \times (D/Y)_{t-1} - (PB/Y) + SFA$$

Each symbol carries specific meaning:

- **D/Y** is the debt-to-GDP ratio: total government debt divided by gross domestic product, the standard way to compare indebtedness across countries of different sizes.
- **r** is the real interest rate the government pays on its debt (the nominal rate adjusted for inflation).

- **g** is the real growth rate of the economy.
- **(r − g)**, the interest-rate-growth differential, is the single most important term in debt mathematics. If the economy grows faster than the government pays interest (g is greater than r), the debt ratio tends to shrink on its own, even with modest deficits, sometimes described as "growing out of debt." If r is greater than g, debt compounds faster than the economy can absorb it, an effect sometimes called a debt snowball.
- **PB/Y** is the primary balance as a share of GDP: government revenue minus spending, excluding interest payments. A primary surplus pulls the debt ratio down; a primary deficit pushes it up.
- **SFA**, the stock-flow adjustment, captures changes in debt that do not appear in the annual deficit, such as bank bailouts or currency valuation effects.

This identity is bookkeeping, not a theory of behaviour, but it is the state equation, the mathematical description of how the system evolves, on which every optimisation framework below is built. Optimisation always needs two ingredients: a state equation describing how the system moves, and an objective function describing what "good" means. This identity supplies the first.

The three sections that follow correspond to three distinct but connected optimisation problems, summarised here before being explained in detail.

Layer	Core question	Mathematical tool	Real-world example
Debt sustainability analysis	Is the debt path likely to remain financeable, given uncertainty about growth, rates, and shocks?	Stochastic simulation (fan charts), probability-based risk zones	IMF's Sovereign Risk and Debt Sustainability Framework
Optimal control of the debt level	When, and how forcefully, should a government intervene to change the debt trajectory?	Stochastic optimal control, Hamilton-Jacobi-Bellman equations, free-boundary thresholds	Ferrari (2018); Cadenillas and Huamán-Aguilar (2016)
Portfolio optimisation of debt structure	Given an existing debt stock, what mix of maturities, currencies, and instruments minimises expected cost for an acceptable level of risk?	Mean-variance and cost-at-risk portfolio optimisation	UK Debt Management Office issuance strategy

Debt sustainability as a constrained, stochastic problem

The most institutionally important application of this mathematics is the IMF's Sovereign Risk and Debt Sustainability Framework (SRDSF), adopted in 2021.[4] It is not a static forecasting exercise; it is explicitly probabilistic. Rather than asking whether debt will be sustainable under a single best-guess forecast, the SRDSF asks what the probability distribution of debt outcomes looks like, given the range of shocks that could plausibly occur. It does this through a fanchart module: thousands of stochastic simulations of the debt path are run, each perturbing growth, interest rates, and the primary balance by amounts drawn from historical volatility, producing a cone-shaped fan of possible debt trajectories rather than a single line.[4] Countries are sorted into risk zones based on the estimated probability that debt is not sustainable, with values below a 20 per cent probability treated as low risk and values above 50 per cent treated as high risk.[5]

A separate calculation, the gross financing needs (GFN) module, addresses a different risk: not whether debt will eventually be repaid, but whether the government can find buyers for the borrowing it needs this year and next, without triggering a funding crisis.[5] This distinction, between solvency risk and liquidity or rollover risk, matters because a country can be arithmetically solvent over forty years and still default within eighteen months, if it cannot refinance maturing bonds during a market panic. Any complete optimisation of government debt must therefore treat *when* debt falls due, not only how much is owed, as a decision variable.

For low-income countries without deep capital-market access, a parallel tool, the LIC Debt Sustainability Framework, run jointly by the IMF and World Bank, classifies countries by debt-carrying capacity (strong, medium, or weak) using a composite indicator of historical performance, growth outlook, remittances, and reserves, and applies different debt-burden thresholds accordingly.[6] A fixed debt ceiling makes little sense across countries with very different capacities to absorb shocks.

The output of this machinery is a constrained optimisation problem: minimise the probability of debt distress, subject to realistic ranges of growth, interest rate, and exchange rate shocks, while keeping financing needs within what markets can plausibly absorb each year.

Debt reduction as an optimal control problem

The IMF's framework measures how much risk a government is carrying. It does not, by itself, identify the mathematically best policy response: how aggressively to run surpluses, and when. That is the domain of optimal control theory, a branch of mathematics originally developed for engineering problems such as guiding rockets, and long applied in economics to problems like Barro's tax-smoothing model, which argued that governments should keep tax rates roughly stable over time, financing temporary spending shocks such as wars or recessions with debt rather than abrupt tax increases.[7]

Recent academic work formalises debt reduction explicitly as a stochastic optimal control problem. Three terms are worth unpacking:

- A **control problem** means a decision-maker, here the government, chooses an action, such as an intervention on the debt ratio, over time to steer a system toward a desired outcome.
- **Stochastic** means the system is affected by randomness. GDP growth, inflation, and market rates fluctuate unpredictably, so the model optimises against a probability distribution of futures rather than a single forecast.
- A **singular control problem**, the variant used in several key papers, is one where the optimal policy is not to adjust the debt ratio continuously but to intervene only occasionally and forcefully: leave the debt ratio alone within a comfortable range, then act sharply once it crosses a threshold. This produces a **free boundary**, a threshold level of debt-to-GDP, derived from the model rather than assumed, above which intervention becomes optimal.

Ferrari modelled exactly this: a government minimising the total expected cost of carrying debt plus the cost of intervening to reduce it, over an infinite time horizon.[8] The tool used to solve such problems is the Hamilton-Jacobi-Bellman (HJB) equation, a differential equation that characterises the minimum achievable cost from any starting debt level, and the policy that achieves it. Ferrari's model treated the debt-to-GDP ratio and the inflation rate as jointly evolving variables, since inflation affects the real value of outstanding debt.

Cadenillas and Huamán-Aguilar derive an explicit formula for the optimal government debt ceiling: an exact mathematical expression, under stated assumptions, for the debt-to-GDP level at which a government should stop letting debt grow and begin actively reducing it.[9] Their model assumes carrying debt imposes a cost that increases faster than debt itself grows (a convex cost), since an extra percentage point of debt-to-GDP is more dangerous at 120 per cent than at 40 per cent, and that reducing debt also carries a cost, since austerity has real economic and political consequences. The optimal ceiling is the point where these two costs balance at the margin.

Brachetta and Ceci build on the foundational work of Domar and Blanchard on debt dynamics, again using stochastic control to characterise optimal fiscal intervention policy.[10,11,12] This literature shares a refusal to treat debt reduction as automatically good in unlimited quantity: the mathematics forces the modeller to state explicitly why debt is costly and how costly reduction itself is, and the optimum emerges from that trade-off rather than being assumed.

Optimising not the level of debt, but its shape

The frameworks above concern how much debt a government should carry and how fast to change it. A separate, and in practice more frequently used, optimisation problem concerns the composition of debt that already exists: its maturity structure, currency mix, and the balance between fixed-rate and inflation-linked instruments. This is the daily work of a sovereign debt management office, and it draws on portfolio theory more than macroeconomics.

The United Kingdom's Debt Management Office (DMO) is a useful case study because its methods and results are published in detail. Its statutory objective is to minimise the long-term cost of financing government borrowing, taking risk into account, while remaining consistent with monetary policy.[13] That phrase, cost taking risk into account, is a portfolio optimisation problem in one sentence, structurally similar to the classic trade-off in mean-variance portfolio theory: for a given expected cost, minimise risk; for a given acceptable risk, minimise expected cost.

This framework is usually called cost-at-risk (CaR). The DMO estimates, under a range of interest-rate and inflation scenarios, both the expected cost of an issuance strategy and how bad the outcome could plausibly become in an unfavourable scenario, at a given confidence level, comparable to Value-at-Risk in finance.[14] Two risks drive the calculation:

- **Refinancing risk**: the danger of having to roll over a large share of maturing debt at once, especially during high interest rates or weak market demand, mitigated by spreading maturities across the calendar and lengthening the average maturity of the debt stock.
- **Interest-rate (duration) risk**: the sensitivity of future interest costs to changes in prevailing rates, which depends on how much debt is short-term or floating-rate versus long-term fixed-rate.

Real figures illustrate the trade-off. At end-2025, UK index-linked gilts, bonds whose principal and coupon payments rise with the Retail Price Index, made up around 25.2 per cent of the wholesale government debt portfolio, roughly £688.5 billion.[13] DMO analysis found that issuing this inflation-linked debt since 1981 saved the government approximately £86.9 billion in direct financing costs at maturity, compared with issuing conventional fixed-rate gilts, largely because investor demand for inflation protection allowed cheaper borrowing.[13] That lower expected cost carried a specific risk: when inflation surged in 2022, the same index-linking mechanically inflated the government's interest bill, a cost that had been dormant for decades. A structure cheaper on average can also be the one that produces the worst outcomes in the worst states of the world.

Maturity choices show the same pattern. The average maturity of the UK's gilt stock fell from around 16.5 years a decade ago to 13.4 years by the end of 2025, as new issuance shifted toward shorter maturities.[15] Shorter debt is typically cheaper to issue on average, since lenders usually demand a smaller premium for shorter-term risk, but it means more of the debt stock needs refinancing sooner, transmitting future interest-rate shocks into the budget faster. This shortening also strengthens the feedback loop between monetary policy and public finances: as more debt refixes quickly, a central bank rate rise reaches the government's own interest bill sooner, which can itself add pressure for fiscal tightening.[15]

Formally, the debt manager's optimisation problem resembles a constrained portfolio problem:

minimise E[Cost], subject to Risk(Cost) ≤ acceptable bound

where "Cost" is the government's future interest bill under a distribution of future rate and inflation paths, and the decision variables are the shares of

debt issued at each maturity and instrument type. A UK parliamentary evaluation of DMO performance noted a genuine limitation of these models: most can only optimise against one measure of risk at a time, even though real debt managers face several distinct risks simultaneously, including refinancing risk, execution risk, liquidity risk, and interest-rate risk, which do not collapse neatly into one number.[16] "Optimal" in this context is always optimal with respect to a simplified model, not optimal in an absolute sense.

Discussion: Where mathematics runs into its limits

Debt optimisation models are unusually prone to being mistaken for more certainty than they can deliver, and the field has a well-documented cautionary tale about exactly this.

Reinhart and Rogoff reported in 2010 that once a country's debt-to-GDP ratio crossed roughly 90 per cent, average economic growth turned sharply negative.[17] The finding was widely cited by policymakers across Europe and the United States as quantitative justification for austerity programmes following the 2008 financial crisis. In 2013, a graduate student, Thomas Herndon, working with economists Michael Ash and Robert Pollin, could not replicate the result. Examining the original spreadsheet, they found a coding error that had excluded several countries from a key average, along with questions about data exclusions and an unconventional weighting method.[18] Once corrected, average growth for countries above the 90 per cent threshold rose from −0.1 per cent to roughly 2.2 per cent, eliminating the sharp growth cliff that had anchored the original argument.[18] Later research has generally found that some negative association between very high debt and growth persists, but the universal 90 per cent cliff does not hold, and debt thresholds are, as Reinhart and Rogoff themselves noted in the original paper, "importantly country-specific."[19]

This episode illustrates three structural limits that apply to every model discussed above, not only that one.

First, model outputs are only as trustworthy as the data and assumptions feeding them. A single spreadsheet error reshaped fiscal policy debates across several countries for years, a reminder that debt mathematics, however elegant, depends on ordinary data hygiene.

Second, optimisation requires an objective function, and objective functions encode value judgements. Whether "cost of debt" is measured purely in interest expense, or also includes the social cost of the spending cuts needed to service it, changes the optimal answer entirely. The mathematics cannot choose the objective; economists and policymakers must, and reasonable people will choose differently.

Third, debt is a political variable as much as a technical one. The IMF has noted that debt projections tend to be systematically optimistic in practice, and that political factors, including the difficulty of committing credibly to future surpluses and electoral cycles, routinely push actual outcomes away from what any optimisation model recommends.[1] A mathematically optimal debt ceiling achieves little without a credible political mechanism to enforce it.

There is also a harder mathematical limitation. Most of the elegant closed-form results in the optimal-control literature, such as explicit debt-ceiling formulas and free-boundary solutions, depend on simplifying assumptions, constant volatility, specific cost functions, a single source of randomness, that make the mathematics solvable but strip out real-world complications such as sudden shifts in investor sentiment, contagion across countries, or political costs that rise sharply only past a certain point rather than smoothly.

Conclusion

Mathematical optimisation does not hand a government a single correct debt number. It offers something more modest but more useful: a disciplined way of making trade-offs explicit. The interest-rate-growth differential clarifies that debt sustainability depends on growth relative to interest costs, not on the size of borrowing alone. The IMF's stochastic sustainability frameworks replace false certainty with honest probability. Optimal control theory shows that deciding when to act is itself a solvable mathematical question, with thresholds that emerge from cost trade-offs rather than political intuition. Portfolio-based debt management shows that even a government carrying a large debt stock still has real optimisation choices, about maturity, currency, and instrument mix, that measurably change its exposure to future shocks, as the UK's £86.9 billion in index-linked savings and its shrinking average maturity both demonstrate.

Optimal government debt is not a fixed destination but a moving, probabilistic target, continuously recalculated as growth, interest rates, and political constraints shift. The mathematics does not remove the need for judgement; it disciplines that judgement, forcing decision-makers to state precisely what they are trading off, and against what range of possible futures. With global public debt projected to pass 100 per cent of world GDP within a few years, that discipline is not an academic luxury.

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Supplementary: Glossary of Key Terms

Term	Meaning
Debt-to-GDP ratio	Total government debt divided by gross domestic product.
Interest-rate-growth differential (r − g)	The gap between the real interest rate a government pays and the real growth rate of its economy.
Primary balance	Government revenue minus spending, excluding interest payments on existing debt.
Stock-flow adjustment	Changes in the debt stock that do not appear in the annual deficit.
Debt sustainability analysis (DSA)	A structured assessment of whether a government's debt path is likely to remain financeable.
Fanchart	A cone-shaped chart showing a range of probable future debt paths from many stochastic simulations.
Gross financing needs (GFN)	The amount of new borrowing a government needs in the near term.
Optimal control theory	A branch of applied mathematics for choosing the best sequence of actions over time to steer a system toward a goal.
Stochastic control	Optimal control under randomness, optimised against a probability distribution of outcomes.
Singular control / free boundary	A control strategy that intervenes only once a variable crosses a threshold derived from the model.
Hamilton-Jacobi-Bellman (HJB) equation	The differential equation used to solve dynamic optimisation problems.
Cost-at-risk (CaR)	A portfolio risk measure estimating how costly a financing strategy could become under adverse scenarios.
Refinancing (rollover) risk	The risk of having to refinance a large share of maturing debt at once.
Duration risk	The sensitivity of future interest costs to changes in prevailing interest rates.

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